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Existence of a capability-equalitarian walrasian equilibrium

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Abstract: According to Sen, the moral standing of the market has to be related to market outcome and inequality rather than to the exercise of people's 'prior' right and efficiency. Although developed within a consequentialist framework, neoclassical economics has usually neglected the problem of inequality in favour of efficiency. Lately, however, a considerable effort has been spent by neoclassical economists to prove that market mechanism satisfies specific fairness requirements. In contrast with this literature, which follows the 'welfarist' method of evaluation of states of affairs, Sen proposes to replace the concept of utility with the concept of capability in assessing inequality. As far as our knowledge is concerned, the question remains open whether the market mechanism is able to ensure in equilibrium an egalitarian resources allocation in terms of capabilities. The aim of this paper is to provide sufficient conditions ensuring the existence of an initial allocation whose walrasian equilibrium ensures the equality of the capability set for every agent in the economy.

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1. Introduction

According to Sen, the moral standing of the market has to be related to results rather than to the exercise of people's 'prior' rights (see, for example, Sen (1989)). Whithin the consequentialist approach the standard neoclassical economic theory has provided some results concerning the optimality of walrasian allocations (see, e.g. Debreu (1959), Arrow and Hahn (1971), Hildebrand and Kirman (1988)). However, in order to assess the market mechanism the crucial issue is the capacity of market in dealing not with optimality but with inequality (Sen (1989, p. 100 ff.)). Although neoclassical economics has usually neglected the problem of inequality, recently a quite considerable amount of work has been carried out within this framework by proving that the market mechanism satisfies specific fairness requirements (see, e.g. Foley (1967), Varian (1974) and, for a useful survey Thomson and Varian (1985)). However, all these attempts follow the 'welfarist' method of evaluation of states of affairs whose weakness has been forcefully pointed out by Sen himself (Sen (1979), (1982)). Sen's solution is to replace the concept of utility with the concept of capability in assessing the inequality among individuals (Sen (1980), (1985), (1988), (1992)).

One of the problems that can be fruitfully analysed by Sen's capability approach is the role of the market mechanism in ensuring an adequate level of well-being, measured in terms of capabilities. Indeed, this problem has been taken up several times by Sen himself (see, e.g. Sen (1985), (1993)) whose attention has been mainly devoted to the efficiency of market equilibria in terms of opportunity-freedoms (using the neoclassical terminology, in terms of the "first welfare theorem"). On this problem Sen has emphasised that the market mechanism can do a good job. Scarce attention, however, has been paid to the efficiency of market equilibria in terms of distributional aspects (i.e. in terms of the "second welfare theorem") on which Sen himself is quite pessimistic, given that implementing a fair allocation may have formidable political as well as informational barriers (Sen, 1993, p. 522). Within this setting, therefore, inequalities arise not only by unfair allocations but also by unequal (and unobservable) endowment in utilization functions (Sen (1993, p. 536)).

As said above, a quite extensive analysis of conditions ensuring the existence of fair allocations (through the market mechanism) has been provided by the neoclassical analysis within a welfarist approach. Accepting Sen's remarks above mentioned concerning the difficulty in implementing fair allocations, at theoretical level the question remains open whether it is possible to ensure, by means of a reallocation of resources, a fair allocation in terms of capabilities. The aim of this paper is to provide sufficient conditions ensuring the existence of an initial allocation whose walrasian equilibrium ensures the equality of the capability set for every agent in the economy. In order to carry out this program we have to define a criterion to compare capability sets. For the sake of simplicity, we shall follow the approach recently adopted by the literature on the ranking of opportunity sets (see e.g. Pattanaik and Xu (1990), Xu (2004)), according to which capability sets are compared on the basis of their "size".

2. The model

Consider an m-good pure exchange non atomic economy *à la* Aumann (1961) and assume that the space of functionings is a k-dimensional Euclidean space; i.e. the well-being is defined by *k focal variables* (see Sen (1992)). The index set of agents is the unit interval I and μ is the nonatomic measure defined on it. Without loss of generality, we assume that $\mu(I) = 1$. Agent $i \in I$ is defined by a triplet (C_i, f_i, h_i) where $C_i \subseteq \Re^m_+$ is the consumption set, f_i: $\Omega \rightarrow S \subseteq \Re^k$ is the "utilization function" which maps good bundles into the person's functionings (see Sen (1981, p. 12)) and h_i: $S \rightarrow \Re$ is a function associating to each functioning a satisfaction level; i.e. h_i is Sen's "happiness" function (*Ibidem*). [1] We assume also that on the set S of agent i's state a Borel measure α is defined. The meaning of this measure will be made precise below. Finally, we assume that the economy is endowed with an initial aggregate endowment vector $\overline{\omega} \in \Re^m_{++}$. Because of this assumption, we can truncate the consumption space of each agent to the compact set $\Omega = \{x \in R^m_+ \mid x_g \leq \overline{\omega}, g = 1, \ldots, m\}$.

Assumption 1. For every $i \in I$, function f_i is continuous.

As a consequence of Assumption 1 and by the compactness of set Ω we obtain that set S can be assumed to be compact. Hence, we can suppose that measure α is σ-finite.

Assumption 2. For every $i \in I$, function h_i is continuous.

Assumption 3. There are T types of agents; i.e. set I can be partitioned into T non-empty subsets, I_1, I_2, …, I_T with $\mu(I_t) > 0$, for every t and such that $f_i = f_j$ and $h_i = h_j$ if and only if $i,j \in I_t$ for $t = 1, 2, \ldots, T$.

Denote by $\mathscr{M}^m$ the set of all measurable functions ω: $I \rightarrow \Re^m$, and set $\mathscr{M}^m_+ = \{\omega \in \mathscr{M}^m \mid \omega(i) \in \Re^m_{++}$ for every $i \in I\}$; i.e. $\mathscr{M}^m_+$ is the set of all strictly positive measurable functions defined on I. An *allocation* is an element ω of $\mathscr{M}^m_+$ such that $\int_I \omega(i) d\mu = \overline{\omega}$. Given Assumption 3, we shall limit

[1] For the sake of simplicity, we skip the distinction, made by Sen (1985) between the characteristic function c: $\Omega \rightarrow \Re^g$ which maps bundle goods into characteristics vectors and the (proper) utilization function F_i: $\Re^g \rightarrow S$ mapping characteristics vectors into functionings. As a matter of fact, we have: $f_i = F_i \circ c$. We are assuming also that each agent has only one utilization function f_i. Moreover, it must be noted that the happiness function h_i can be replaced by the "valuation function" v_i.

ourselves to allocations ensuring equal endowments to agents of each type. Thus, a *type allocation* is an element ω of $\mathscr{M}^m_+$ such that $\int_I \omega(i) d\mu = \overline{\omega}$ and $\omega(i) = \omega(j)$ whenever $i, j \in I_t$ for $t = 1, 2, \ldots, T$. Denote by $\mathscr{T}$ the set of all possible type allocations

Choose an initial type allocation $\omega \in \mathscr{T}$ and consider a price vector $p \in \Delta_{++}$, where Δ_{++} is the (m-1)-dimensional strictly positive simplex. Agent i faces a budget set $B(p, \omega(i))$, and therefore the associate set of functionings $f_i(B(p, \omega(i)))$, among which he/she chooses the "best" one $s_i(p, \omega(i))$.Clearly, the mapping B: $\Delta_{++} \times \mathscr{T} \to \Re^m{}_+$ as previously defined is Sen's entitlement mapping (see, e.g., Sen (1981)). Clearly, $s(i)(p, \omega(i))$ is the solution to the following programme:

$$\text{Max } h_i(s(i)) \quad \text{s.t. } s(i) \in f_i(B(p, \omega(i))). \qquad (*)$$

The optimal *good* bundles are those bundles whose images under f_i are vectors in $s(i)(p, \omega(i))$; i.e. they are the elements in the set $b(i)(p, \omega(i)) = f_i^{-1}(s(i)(p, \omega(i)))$. Hence, the excess demand of agent i is set: $z(i)(p, \omega(i)) = b(i)(p, \omega(i)) - \omega(i)$.

Following Sen's approach we interpret set $f_i(B(p, \omega(i)))$ as the (market) *capabilities* of agent i having an allocation $\omega(i)$ and facing a price vector p. Therefore, following a particular formalisation of Sen's capability approach according to which the "size" of the capability set is a measure of well-being (see the end of Section 1), the number $\alpha(f_i(B(p, \omega(i)))$ can be naturally interpreted as a measure of welfare of agent i when endowed with an endowment vector $\omega(i)$ and facing a price vector p.[2]

A *walrasian equilibrium* at type allocation ω (a *WE-*ω) is a price vector $p^* \in \Delta_{++}$ such that $\int_I z(i)(p^*, \omega(i)) = 0$. A *WE-*$\omega$ is *capacity-equalitarian* if $\alpha(f_i(B(p^*, \omega(i)))) = \alpha(f_j(B(p^*, \omega(j))))$ for every $i, j \in I$.[3]

Assumption 4. For every $\omega \in \mathscr{T}$ if there exists a *WE-*ω, then it is unique.

This condition is ensured, for example, if the economy satisfies the gross substitutability condition (see e.g. Hildebrand and Kirman (1988)). Denote by $p(\omega)$ the (unique) walrasian equilibrium price associated with the allocation ω.

Theorem. Under Assumptions 1-4 there exists a type allocation $\omega^* \in \mathscr{T}$ such that the associated *WE-*ω^* is capacity equalitarian.

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[2] As α is a Borel measure and $f_i(B(p, \omega(i)))$ is always compact, then $\alpha(f_i(B(p(\omega), \omega(i)))$ is well defined.

[3] In this paper we do not consider the survival problem (see e.g. Sen (1981)).

In order to prove the theorem we have to prove several preliminary results.

Lemma 1. Under Assumptions 1 2 and 3, for every $\omega \in \mathscr{T}$ there exists a *WE*-ω.

Proof. For every $\omega \in \mathscr{T}$, correspondence B: $\Delta_{++}\times \mathscr{T} \to \Re^{m}{}_{+}$ defined by set $B(p, \omega(i)))$ is compact-valued and continuous with respect to p. As a consequence, by Assumption 1, for every $\omega \in \mathscr{T}$, correspondence f_j: $\Delta_{++}\times \mathscr{T} \to S$ defined by $f_j(B(p, \omega(j)))$ is compact valued and continuous with respect to p. By continuity, programme (*) is well defined and a solution $s(i)(p, \omega(i))$ indeed exists. Moreover, given $\omega \in \mathscr{T}$, correspondence s_i: $\Delta_{++}\times \mathscr{T} \to S$ defined by $s_i(p, \omega(i))$ is closed and upper hemi-continuous with respect to p, by Berge Theorem (see Berge (1980)). Moreover, since functions f_i are continuous and S is compact (see remark after Assumption 1), it follows that $f_i^{-1}(\cdot)$ is upper hemi-continuous (see Fitzroy (1974, Lemma 4)); hence, correspondences $b(i)(p, \omega(i))$ and, therefore, $z(i)(p, \omega(i))$ are also upper hemi-continuous with respect to p. By using standard arguments for large economies (see e.g. Hildebrand (1974)), we obtain the claim as $\omega(i) \in R^{m}{}_{++}$ for every $i \in I$.

Lemma 2. Under Assumnptions 1, 2, 3 and 4, the walrasian function $p(\omega)$ is a continuous function from $\mathscr{T}$ into Δ_{++}.

Proof. Under Assumptions 1, 2 and 3, it is a standard result that the set of walrasian equilibria is a upper hemi-continuous correspondence. Assumption 4 implies the assertion.
Consider the function α_i: $\mathscr{T} \to R$ defined as follows: $\alpha_i(\omega) = \alpha(f_i(B(p(\omega), \omega(i))))$. Function α_i associates to each type allocation the "size" of the capability set of agent i.

Lemma 3. Functions α_i are continuous functions mapping set $\mathscr{T}$ into $\Re$.

Proof. Take any measure μ such that $\alpha << \mu$. Since measure α is σ-finite (see the remark after Assumption 1), then by the Radon-Nikodym Theorem there is a summable function γ such that $\alpha(A) = \int_A \gamma\, d\mu = \int_S \gamma\, \chi_A d\mu$ for every set $A \subseteq S$, where χ_A is the characteristic function. Consider now any convergent sequence $\{\omega_n\}$ in $\mathscr{T}$. It defines a sequence of T compact subsets in S, $\{(f_i(B(p(\omega_n), \omega_n(i))))\} \equiv \{(Z_i(\omega_n))\}$. Thus, for every i, $\alpha(Z_i(\omega_n)) = \int_{Z_i(\omega_n)} \gamma\, d\mu = \int_S \gamma \cdot \chi_{Z_{i(\omega n)}} d\mu$. By definition, $\gamma \cdot \chi_{Z_{i(\omega n)}} \leq \gamma$, moreover, the limit of $\gamma \cdot \chi_{Z_{i(\omega n)}}$ exists for every $s \in S$. Hence, by the Lebesgue Dominated Convergence Theorem $\int_S \lim_n(\gamma \cdot \chi_{Z_{i(\omega n)}}) d\mu = \lim_n \int_S \gamma \cdot \chi_{Z_{i(\omega n)}} d\mu$; i.e. $\alpha(Z_i(\lim_n \omega_n)) = \lim_n \alpha(Z_i(\omega_n))$.

Set $\mu_{\min} = min\ [\mu(I_t)|\ t = 1, 2, \ldots, T\ ]$. By assumption $\mu_{\min} > 0$. Consider now a vector $\varepsilon \in R^m_{++}$ and define the allocations $\omega_{(q)} = \{\omega \in \mathscr{X}\ |\omega(i) = \mu_{\min} \cdot \varepsilon$ if $i \in I_t$ with $t \neq q$, and $\omega(i) = \left(\overline{\omega} - \varepsilon\left(1 - \mu\left(I_q\right)\right)\right)\mu\left(I_q\right)$ if $i \in I_q\}$, where $q = 1, 2, \ldots, T$.

Remark 1. From Lemma 3 it is possible to find $\varepsilon \in \Re^m_{++}$ such that $T \cdot \mu_{\min} \cdot \varepsilon < \overline{\omega}$ and such that for every $t = 1, 2, \ldots, T$: $\alpha_i(\omega_{(q)}(i)) > \alpha_j(\omega_{(q)}(j))$ whenever $i \in I_q$ and $j \notin I_q$.

In the following result CH indicates the convex hull operator.

Lemma 4. Let $f: X \subseteq \Re^{m \times n}_{++} \rightarrow Y \subseteq \Re^q$ be a continuous function. If there exist n strictly positive matrices $X_1^*, X_2^*, \ldots, X_q^*$ in X such that $f_i(X_i^*) > f_j(X_i^*)$, $i, j = 1, 2, \ldots, q$ and $j \neq i$, then there exists a matrix $X^\circ \in CH\{X_1^*, X_2^*, \ldots, X_q^*\}$ such that $f_i(X^\circ) = f_j(X^\circ)$, $i, j = 1, 2, \ldots, q$.

Proof. Take any $X \in CH\{X_1^*, X_2^*, \ldots, X_q^*\}$. Clearly, there exists q real numbers $\beta_1(X)$, $\beta_2(X)$, $\ldots$, $\beta_q(X)$ such that $\beta_j(X) \geq 0$, $j = 1, 2, \ldots, q$, and $\sum_{j=1}^{n} \beta_j(X) = 1$ such that $X = \sum_{j=1}^{q} \beta_j(X) X_j^*$. Now define the mapping Z: $CH\{X_1^*, X_2^*, \ldots, X_q^*\} \rightarrow CH\{X_1^*, X_2^*, \ldots, X_q^*\}$ as follows:

$$Z(X') = \sum_{j=1}^{q} \beta_j(X') X_j^*,$$

where $\beta_j(X') = \beta_j + \left(\frac{\sum_{j=1}^{q} f_j(X')}{n} - f_j(X') \right)$.

The image of Z is in $CH\{X_1^*, X_2^*, \ldots, X_q^*\}$ as:

$$\sum_{j=1}^{q} \beta_j(X') = \sum_{j=1}^{q} \beta_j + \sum_{j=1}^{q} \left(\frac{\sum_{j=1}^{n} f_j(X')}{q} - f_j(X') \right) = \sum_{j=1}^{q} \beta_j = 1.$$

By continuity, there exists a fixed point $X^{**} = Z(X^{**})$. As $X^{**} = \sum_{j=1}^{q} \beta_j(X^{**}) X_j^*$, then, it follows that $\sum_{j=1}^{q} \beta_j(X^{**}) X_j^* = \sum_{j=1}^{q} \left(\beta_j(X^{**}) + \left(\frac{\sum_{j=1}^{q} f_j(X^*)}{q} - f_j(X^{**}) \right) \right) X_j^*$. As X_j^*s are strictly positive matrices, it follows that $\frac{\sum_{j=1}^{q} f_j(X^*)}{q} - f_j(X^{**}) = 0$ for every $j = 1, 2, \ldots, q$. This concludes the proof.

For each q = 1, 2,…, T, consider the m×n-dimensional matrix $\omega'_{(q)}$ whose q-th column is the vector $\left(\overline{\omega}-\varepsilon\left(1-\mu\left(I_q\right)\right)\right)\mu\left(I_q\right)$ and the remaining m -1 columns are equal to $\mu_{\min}\cdot\varepsilon$. By Remark 1, these matrices are all strictly positive and each of them uniquely defines a possible type allocation.

Proof of Theorem. Take $i_t \in I_t$, t = 1,2, …, T and consider for the moment the fictitious economy made only by agents $\{i_1, i_2, \ldots, i_T\}$. An allocation of this economy is a matrix W whose t-th column is the endowment vector of agent i_t. Denote by $W(i_t)$ the matrix whose columns are equal to vector $\mu_{\min}\cdot\varepsilon$ except the i_t-th column which is equal to $\left(\overline{\omega}-\varepsilon\left(1-\mu\left(I_{i_t}\right)\right)\right)\mu\left(I_{i_t}\right)$. By construction, $\alpha_{it}(W(i_t)) > \alpha_{it'}(W(i_{t'}))$, $t' \neq t$. Hence, by Lemmas 3 and 4 there exists a m×T nonnegative matrix $W^*=(\omega^*_{i_t}) \in CH\{W(i_1), W(i_2), \ldots, W(i_T)\}$ such that $\alpha_{it}(W^*)= \alpha_{it'}(W^*)$, $t' \neq t$. Consider now the allocation ω^* for the original economy defined as follows: $\omega^*(i) = \omega^*_{i_t}$ if $i \in I_t$. By construction, $\omega^* \in \mathscr{A}$ and moreover one has that: $\alpha_i(w^*(i)) = \alpha_j(w^*(j))$ for every i, j = 1, 2, …., T. This completes the proof.

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