Ranking opportunity sets in the space of functionings

Francesco Farina*
University of Siena

Eugenio Peluso
THEMA-Université de Cergy-Pontoise and University of Siena

Ernesto Savaglio
University of Siena

July, 2003 (THIS VERSION)

Abstract

In this paper we define and rank opportunity sets in the space of individual functionings. A microeconomic foundation of our analysis is provided by the Sen's functionings' theory. We suppose a target vector in the evaluative space, and we define in a novel way some properties of the opportunity sets as *essentiality* and *freedom*. As a main result, we give an axiomatic characterization of the ranking induced by the Euclidean distance between the opportunity sets and the target.

Keywords: Opportunity sets, functionings, target, set inclusion, essentiality, freedom.

JEL classification: D31, D61, I31.

1 Introduction

Sen defines "the functionings" of an individual in terms of "the command over the characteristics of goods", that is the extent to which the individual succeeds in using the various characteristics of the goods he has access to (Sen, [8]). In this paper, we construct a theoretical framework to compare and rank individual opportunity sets as combinations of functionings.

The problem of ranking opportunity sets has been discussed, among others, by Pattanaik and Xu, who in their seminal 1990 paper [3] introduced an

*Corresponding author. We acknowledge useful discussions with W. Gaertner and useful suggestions of two anonimous referees. E. Savaglio thanks THEMA for hospitality in Spring 2002. We are grateful for financial support provided by the M.I.U.R. The usual disclaimer applies.

ordering across opportunity sets based on cardinality. Their approach has subsequently been generalized by Klemisch-Ahlert [2]. More recently, Pattanaik and Xu [4] have studied how "freedom of choice" depends on diversity across the elements composing the opportunity sets, and Xu [11] has considered the problem of ranking opportunity sets that are linear budget sets in the space of functionings. An alternative approach has been developed by Sen [7], [6], who assigns a central role to individual preferences on the elements of a set in the evaluation of opportunities provided by it.[1]

In this paper, we do not study any "indirect ranking" of opportunity sets based on individual preferences. However, we are interested in stressing the different conditions of life and the link between some individuals' attributes and their opportunities. Since the capacity of transforming goods in personal realizations depends on the individual process of consumption, various circumstances can affect the individuals' ability to use some commodities, as observed by Sen: *"There may be some accentuation of inequality due to the 'coupling' of i) economic inequality and ii) unequal advantages in converting incomes into capabilities, the two together intensifying the problem of inequality of opportunity-freedoms"* (Sen,[7],p.536)..."D*ifferences in age, gender, talents, disability etc. can make different persons have quite divergent substantive opportunities even when they have the very same commodity bundle"*(Sen, [1],p.209). The empirical evidence supports the above conjectures. As for the UN Human Development Index, which cares about a decent standard of living, a long and healthy life and education, we explore the relationship between the income level and the two functionings health and education in Italy and in Germany,[2] and the role of objective conditions of life (such as the area of residence) in affecting the individuals' consumption technology. A clear correlation between income and functionings appears (see Table 1 in the Appendix 2). In both countries there is a higher relative incidence of low income individuals - those belonging to the first two quintiles - for very low education and bad health (E3-H2) and very low education and very bad health (E3-H1). Conversely, there is a higher relative incidence of high income individuals - those belonging to the last two quintiles - for good education and good health (E1-H3). Moreover, by controlling for different areas of residence, the consumption technology (in terms of production of both health and education) of individuals at the same quintile appears to be affected by this source of diversity of conditions of life. In Italy, the first two quintiles show higher percentages of individuals with the worst education and health conditions in the South than in the North. In fact, the worst couple of functionings (E3-H1), and in addition also E2-H3 and E3-H3, present opposite profiles of income distribution in the most advanced and the backward areas, respectively. As for Germany, a positive correlation between high education and high income is especially apparent in the West. Similarly to Italy, higher percentages of low education and good health (E2-H3) can be found in the back-

[1] For an overview of the literature, see Sugden [9].

[2] Our empirical analysis is performed on data collected by the European Community Household Panel for Italy and Germany, as these two countries represent a large sample with a sharp polarisation across advanced and backward areas.

ward area for the same quintiles. Therefore, a strong correlation between income and functionings, and different conditions of life inducing different consumption technologies - are both confirmed by empirical analysis.

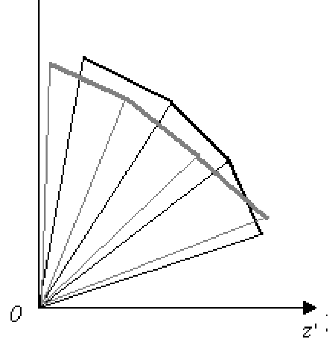
In the following, we concentrate attention on a specific aspect of such a large problem, providing an answer to the following question: "How can we evaluate different opportunities in terms of functionings, when individuals have the same income?". Given a finite number of individuals types and for a fixed income, we use the Sen's functionings'theory to obtain compact, convex and comprehensive opportunity sets in the space of functionings. Then, we develop a theoretical framework which allows establish an ordering over general families of opportunity sets.We agree with the relevant literature in considering the set inclusion principle (a set offers more opportunities than its subsets) as the most intuitive and acceptable criterion to compare two opportunity sets. Yet, set inclusion turns out to represent a too weak criterion, as it is silent whenever two sets mutually intersect or are disjoint. Therefore, we assume the existence of a bliss point (a target) in the space of functionings and provide an axiomatic structure which allows us to rank different opportunity sets with respect to this target. The axioms we adopt are *Set Inclusion*, *Essentiality*, *Freedom, Contraction* and *Scale Independence*. Two features of the axiomatic structure are worth noting. First, our framework introduces the *Essentiality* axiom in an original way: a vector of functionings is evaluated essential (or not) with respect to the achievement of a given target. Second, the concept of freedom proposed here consists in the availability of a certain number of options in the opportunity set, allowing for a certain degree of liberty of choice with respect to a vector of functionings representing the best approximation to the goals defined by the target. The ranking that we axiomatically characterize is strictly related to the complete order induced by the Euclidean distance between the opportunity sets and the target.

The paper is organized as follows. In section 2, we draw on the Sen's functioning model the problem of the comparison of compact, convex and comprehensive opportunity sets. In Section 3, we develop an axiomatic model aimed at comparing different individual opportunity sets. Section 4 concludes, also by providing some hints for further developments.

2 A linear functionings model

A functioning is the extent to which the individual is able to transform the goods' characteristics in personal realizations. We distinguish between the possession of a good and the ability to do all can be done with this good. More precisely, let $\mathbb{R}^G_+$ and $\mathbb{R}^C_+$ be the Euclidean spaces of G goods and C functionings, respectively. Given a population of N individuals, we suppose a finite number $I \leq N$ of individual types. By considering a linear version of the Sen functionings model (Sen, [8], pp.6-7), consumers purchase and use goods in combinations minimizing the 'costs of production' of the desired functionings. According a twice linear technology, goods are tranformed into characteristics and characteristics into a functionings vector. For each type $i \in I$, we define $\mathbf{b}_g^i \in \mathbb{R}^C$ the vector that represents the maximal level of functionings obtained by an individual of type i consuming one unit of good g. In the case of two functionings, z_1^i and z_2^i, produced by four goods: $\mathbf{q} = q_1, ., q_4$ with prices $p_{1,}..p_4$, the efficient frontier in the space of functionings is given by the boundary of the set obtained by joining the vectors $\mathbf{z}_g^i = \left(\frac{y}{p_g} b_{1g}^i, \frac{y}{p_g} b_{2g}^i\right)$, with $g = 1, ..., 4$. These vectors of functionings are those obtained by the individual of type i spending all his (or her) income in the good g, with $g = 1, ..., G$. In Figure 1, we draw the efficient frontiers in the functionings space of two different individuals for a given income.

Fig.1



We define $A^i = co(0, \mathbf{z}_g^i | g = 1, ..., G)$ the convex hull generated by the vector 0 and the vectors $\mathbf{z}_g^i$. We admit that free disposal is allowed, in the sense that, given a feasible vector of functionings $\mathbf{z} \in A^i$ and a vector $\mathbf{z}' \leq \mathbf{z}$ (that is, every component of $\mathbf{z}'$ is less than or equal to the corresponding component of $\mathbf{z}$), we assume that $\mathbf{z}' \in A^i$. In other terms, we allow production plans $\mathbf{z}'$ generating an equal or smaller amount of functionings by using at least as much of all inputs than in the production of $\mathbf{z}$.[3] As a consequence, we interpret the set A^i and the subset of $\mathbb{R}_+^C$ "below it" as the opportunity set in the space of functionings of an individual of type i, for a fixed level of income. By linearity, these sets are compact and convex. The property induced by free disposal assumption is named "comprehensiveness" and it will be formally stated in the next section. As a further consequence of linearity, observe that the distance from the origin of the functionings vector produced by an individual of type i who only buys good g is given by: $\frac{y}{p_g} \left\|\mathbf{b}_g^i\right\|$, where $\|\cdot\|$ is the *Euclidean norm.* Such a distance is linear in income. [4] Then, for given prices and consumption technology, an increase in the income level produces a proportional expansion of the functionings' efficient frontier. In the next section, we develop a general

[3] For instance, the same person can read a book with a different degree of attention or cook a dish of spaghetti with different care, obtaining, of course, different results.

[4] The ratio $p_g / \left\|\mathbf{b}_g^i\right\|$ can be considered as *subjective price* of good g for an individual of type i. The market price is adjusted by the individual productivity in consumption, so that for people of type i, a loss in consumption efficiency has the same effect than an increasing in market price.

model in order to rank opportunity sets in the space of functionings, which can be used in order to compare the class of opportunity sets presented above.

3 Ranking opportunity sets

3.1 The axiomatic structure

Let $\mathbb{V}$ a collection of I compact, convex and comprehensive opportunity sets belonging to $\mathbb{R}^C_+$. We denote $A, B, ...$ the elements of $\mathbb{V}$. We recall that a set $B \in \mathbb{R}^C_+$ is comprehensive if for any $\mathbf{b} \in B$, $\mathbf{b}' \leq \mathbf{b}$ implies $\mathbf{b}' \in B$. Our aim is to define an order $\succsim$ on the elements of $\mathbb{V}$. We respectively define $\succ$ and $\backsim$ the asymmetric and symmetric part of $\succsim$. When $A \succsim B$, we say that A provides at least as opportunities as B.

In order to adapt the notion of set-inclusion in this set up, we introduce the following:

Definition 1 *$B \subset A$ if for all $\mathbf{b} \in B$ there exists a vector $\mathbf{a} \in A$ such that $a_j > b_j$ for $j = 1, ..., C$.*

$B \subseteq A$, if for all $\mathbf{b} \in B$ there exists a vector $\mathbf{a} \in A$ such that $a_j \geq b_j$ for $j = 1, ..., C$, with $>$ for at least a j.

The following property is generally accepted. It reflects the idea that a set contents more opportunities than its subsets:

Axiom 1 (I) *(Inclusion). For all $A, B \in \mathbb{V}$,*

$$B \subset A \Longrightarrow A \succ B.$$

The problems arise when two sets are not comparable by Axiom I. In such a case, we consider a reference point $\mathbf{t}$, a *target* in the space of functionings. We suppose that the target is exogenous: it can be a 'mean level' of functionings fixed by a social observer or a sort of 'poverty line'.

The main idea on which we found a target-based ranking $\succsim$ is that, given a target $\mathbf{t}$, not every point of an opportunity set could be evaluated as important. This principle is formally expressed by the notion of *essentiality.*

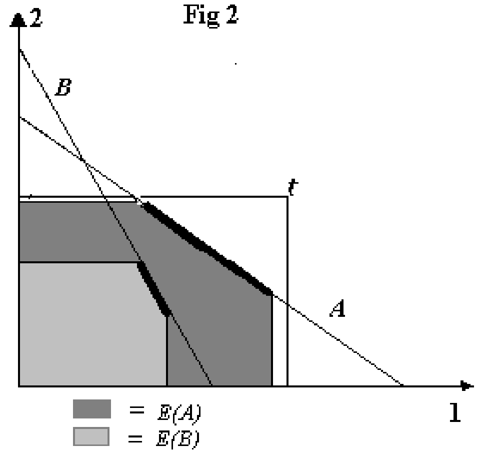
Axiom 2 (E) *(Essentiality wrt a target). For a given $\mathbf{t} \in \mathbb{R}^C_{++}$ and for all $A \in \mathbb{V}$, whenever $\mathbf{t}$ is not strictly included in A, there exists a compact, convex and comprehensive set $E(A) \subseteq A$ such that:*

1. $\mathbf{x} \leq \mathbf{t}$ $\forall \mathbf{x} \in E(A)$,

2. $E(A) \backsim A$

Notice how Axiom E differs from the notion of essentiality proposed by Puppe [5]. Here the essentiality of a vector of functionings is determined by the relation between such a vector and the 'social target'. For instance, if a vector contains a higher quantity of some functioning than $\mathbf{t}$, it cannot be essential (imagine, for instance, a level of education equal to a Ph.D.). Moreover, according to our definition, the notion of essential set is meaningful just for opportunity sets which do not include the target as an interior point. In fact, there is not reason to 'shrink' very large opportunity sets considering them as equivalent to the vector $\mathbf{t}$. This also guarantees that Axiom E is independent of Axioms I. Observe that if the relation $\succsim$ satisfies Axiom I, the set $E(A)$ lies on the boundary of A.

Using the notions of set inclusion and essentiality, it is possible to compare some opportunity sets as shown in the figure 2 below.



The crossing between opportunity sets A and B occurs for a level of functioning 2 evaluated as uninteresting. On the contrary, over the *essential* part of individual functionings, the opportunity set A is clearly larger than B. The intuitive idea of *inclusion between essential sets* allows in this case the ranking of a couple of opportunity sets that the set inclusion principle could not rank. On the other hand, without introducing other properties of the essential set, it is easy to find counterexamples for which the converse is true. In the following, we introduce further axioms which allow us to give an operational content to the intuition sketched in the previous example. For this purpose, we need some geometrical tools. First, denoted $\|\mathbf{a}-\mathbf{t}\|$ the Euclidean distance between vectors $\mathbf{a}$ and $\mathbf{t}$, we provide a precise definition of *distance* between a point and an opportunity set.

Definition 2 *For any $A \subseteq \mathbb{V}$, the Euclidean distance $d(A, \mathbf{t})$ between A and the vector $\mathbf{t} \in \mathbb{R}^C_{++}$ is defined by:* $\inf\{\|\mathbf{a}-\mathbf{t}\|, |\mathbf{a} \in A\}$,

We introduce the notion of *support hyperplane.*

Definition 3 *Let $A \in \mathbb{V}$ and H respectively be a opportunity set and a hyperplane. Then H is a support hyperplane of A, if H meets A and A lies in one of the closed half-spaces determined by H.*

In what follows, we provide in more detail the geometric intuition that, given a target $\mathbf{t}$ and an opportunity set A, there exists a point $\mathbf{a}^*$ of A which is the *nearest* one to $\mathbf{t}$. In particular, $\mathbf{a}^*$ is unique and it lies on the boundary of A. Moreover, next lemma shows that there exists a hyperplane H, which supports A in $\mathbf{a}^*$ and is orthogonal to the line joining $\mathbf{t}$ and $\mathbf{a}^*$.

Lemma 1 *Let $A \in \mathbb{V}$ be an opportunity set and let $\mathbf{t}$ be a target point external to A. Then there exists a unique point $\mathbf{a}^*$ of A, belonging to the boundary of A, such that:*

$$d(\mathbf{a}^*, \mathbf{t}) = \inf \{\|\mathbf{a} - \mathbf{t}\| \, | \mathbf{a} \in A\}$$

holds. Moreover, through $\mathbf{a}^*$ passes a hyperplane H which supports A and is orthogonal to the line $\lambda\mathbf{t} + (\mathbf{1}-\lambda)\mathbf{a}^*$.

Proof. See Appendix. ∎

If all individuals of a society should converge towards the target, which represents a desired collection of functionings, the lower is the distance of each individual opportunity set from $\mathbf{t}$, the higher is the 'degree of efficiency of it. For instance, in the model presented in the previous section, a lower amount of income is needed to reach the target. We complete the notion of essentiality by an axiom conveying the idea that individual achievements on the space of functionings have to be warranted by a minimal content of "freedom of choice". We introduce this idea by a picture that describes the case of an opportunity set H in $\mathbb{R}^3_+$, consisting of a compact and convex portion of a hyperplane.

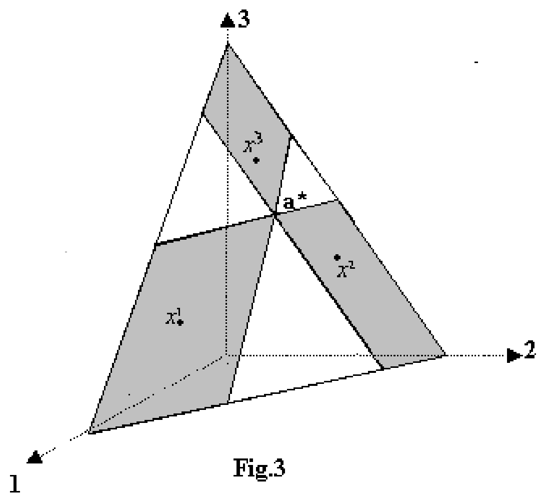


Fig.3

The projection $\mathbf{a}^*$ of the target on H represents the point of the opportunity set 'more efficient in approaching the target'. We require that the essential set of H has a weak property of symmetry with respect to $\mathbf{a}^*$. In particular, we suppose that for each functioning $z = 1, 2, 3$ the essential set $E(H)$ contains at least a vector with a higher level of the functioning z than the quantity a^*_z contained in the vector $\mathbf{a}^*$. We can imagine that this vector is obtained from the point $\mathbf{a}^*$ by renouncing to the production of *all* the functionings different of z. In the figure above, the essential set must contain three vectors as $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$, such that the quantity of the functioning 1 in $\mathbf{x}^1$ is higher than a_1^* and the converse is true for the functionings 2 and 3. Similarly, vector $\mathbf{x}^2$ has $x_2^2 > a_2^*$, $x_1^2 < a_1^*$ and $x_3^2 < a_3^*$. Finally, the vector $\mathbf{x}^3$ has $x_3^3 > a_3^*$, $x_1^3 < a_1^*$ and $x_2^3 < a_2^*$.

More generally, we state the following:

Axiom 3 (F) *(Freedom) Suppose an opportunity set H given by a positive, compact and convex portion of the hyperplane of equation $h_0 + \mathbf{h} \cdot a = 0$, with $h_i \geq 0$ for $i = 1, ..., C$. Let $\mathbf{a}^* = (a_1^*, ..., a_C^*)$ be the projection of an exterior target $\mathbf{t}$ on H. Then $E(H)$ must contain vectors $\mathbf{x}^z = (x_1^z, ..., x_C^z)$, with $z = 1, ..., C$, such that the components $x_j^z > a_j^*$ if $j = z$ and $x_j^z < a_j^*$ if $j \neq z$.*

In order to give an interpretation of the Axiom F, imagine that there is a factor used in the production of each functioning, e.g. leisure (for a given income). We suppose that it is possible to reduce the time spent in all the activities which produce functionings different from a given one - say, health - in order to spend this time in activities improving the personal health level. Obviously, the same could be done for education or other functionings. Axiom F means that it is considered as essential the fact that each type of individual has the possibility of moving away a little bit from the more "efficiency oriented" behavior, increasing the production of the functioning she likes more. In the following remark, we stress that our main results still hold under such a stronger notion of freedom.

Remark 1 *The condition requiring $x_j^z > a_j^*$ if $j = z$ and $x_j^z < a_j^*$ if $j \neq z$ in Axiom F could be replaced by $x_j^z > \theta_j a_j^*$ for some $\theta_j > 1$ if $j = z$ and $x_j^z < a_j^*$ if $j \neq z$. In this case, higher 'degrees of freedom' could be guaranteed, and with a different intensity for each functionings.*

The next axiom is a technical one, similar to the Nash's Independence of Irrelevant Alternatives. It requires that if a vector of functionings $\mathbf{a}$ belongs to the boundary of the essential set of A, and $\mathbf{a}$ belongs to the subset B of A, then the essential set of B must contain $\mathbf{a}$.

Axiom 4 (C) *(Essentiality Contraction) Given $\mathbf{a} \in E(A)$, such that $\nexists \mathbf{a}' \in A$ with $\mathbf{a}' > \mathbf{a}$, for any $B \subseteq A$, if $\mathbf{a} \in B$ then $\mathbf{a} \in E(B)$.*

Given a linear relation between opportunity sets and income as assumed in in the previous section, we could require that the ranking between two opportunity sets has not to be affected by a reduction of the income chosen as reference. In a more general way, we introduce the following axiom:

Axiom 5 (S) *(Scale Independence) Let A, $B \in \mathbb{V}$. Suppose that $A \succ B$. Then for any $\alpha \in (0, 1)$, we have $\alpha A \succ \alpha B$.*

The asymmetric part of $\succsim$ is invariant to proportional "shrinking" of opportunity sets. We do not require such an invariance property also for the case of "expansions" of opportunity sets. In fact, for very high levels of incomes, any type of individual could reach the target.

3.2 The minimal distance order

The complete order that we will characterize using the previous axioms is denoted by $\succsim_{\min}$. Such a relation ranks the opportunity sets according to their distance from the target, with an important exception. In fact, if both opportunity sets contain the target as an interior point, their distance from the target is zero, and they could be considered as equivalent. On the contrary, we impose $\succ_{\min}$ in case of strict set inclusion and $\backsim_{\min}$ otherwise. Formally:

Definition 4 *For any $A, B \in \mathbb{V}$, we define the relation $\succsim_{\min}$ as follows:*

$$A \succsim_{\min} B \text{ if } d(A, \mathbf{t}) \leq d(B, \mathbf{t}) \quad \text{whenever } \mathbf{t} \notin A \cap B$$

$$\left.\begin{array}{ll} A \succ_{\min} B & \text{if } B \subset A \\ A \backsim_{\min} B & \text{otherwise} \end{array}\right\} \text{whenever } \mathbf{t} \in A \cap B.$$

The next lemma shows under which axioms the minimal distance point of an opportunity set from the "target" must belong to the essential set.

Lemma 2 *If the relation $\succsim$ on $\mathbb{V}$ has essential sets which satisfy Axiom F and $\succsim$ satisfies Axioms I, C, E, then, for any $A \in \mathbb{V}$, we have $\mathbf{a}^* \in E(A)$.*

Proof. See Appendix. ■

Finally, in the next theorem we provides a characterization of $\succsim_{\min}$.

Theorem 1 *Suppose that $\succsim$ on $\mathbb{V}$ has essential sets which satisfy Axiom F. Then $\succsim$ satisfies Axioms I, E,C, S if and only if $\succsim = \succsim_{\min}$.*

Proof. See Appendix ■

4 Conclusions and further extensions

Our analysis is posited in the recent strand of literature which attempts to overcome the traditional consideration of income as the sole source of inequality across individuals. In fact, we have constructed an axiomatic framework such that different individual opportunity sets of functionings, for a given the income level, can be ranked by the distance from a reference point. In order to render operationally useful our theoretical model, some improvements are in order. First of all, the qualitative analysis of the relationship between the income level on the one side, and the functionings on the other side, should be ameliorated by resorting to indicators for the functionings which are not exposed to the risk of producing misleading results. The rigorous measurement of the level of functionings would also make more meaningful the use of the distance between the opportunity sets and the social target as the instrument of comparison of the opportunity sets. Finally, the central question remains of how to give rigorous foundations to the consideration of an unique target which applies to all individuals in society. The UN Human Development Index is just an example of the increasing efforts that the international community is devoting to a precise assessment of precise standards functionings.

References

[1] J. Foster and A. Sen. On economic inequality after a quarter century. In *On Economic Inequality (Expanded Edition)*. Clarendon Press, Oxford, 1997.

[2] M. Klemisch-Ahlert. Freedom of choice : A comparison of different ranking of opportunity sets. *Social Choice and Welfare*, 10:189–207, 1993.

[3] P. Pattanaik and Y. Xu. On ranking opportunity sets in terms of freedom of choice. *Recherches économiques de Louvain*, 56:383–390, 1990.

[4] P. Pattanaik and Y. Xu. On diversity and freedom of choice. *Mathematical Social Sciences*, 40:123–130, 2000.

[5] C. Puppe. An axiomatic approach to preference for freedom of choice. *Journal of Economic Theory*, 68:174–199, 1996.

[6] A. Sen. Welfare, preference and freedom. *Journal of Econometrics*, 50:15–29, 1991.

[7] A. Sen. Markets and freedoms: Achievements and limitations of the market mechanism in promoting individual freedoms. *Oxford Economic Papers*, 45:519–541, 1993.

[8] A. Sen. *Commodities and Capabilities*. Oxford university press, Oxford, 1999.

[9] R. Sudgen. The metric of opportunity. *Economics and Philosophy*, 14:307–337, 1998.

[10] R. Webster. *Convexity*. Oxford University Press, Oxford, 1994.

[11] Y. Xu. Functionings, capabilities and the standard of living:an axiomatic approach. *Economic Theory*, 20 (2):387–399, 2002.

5 Appendix

Proof of Lemma1

We know that $\mathbf{a}^*$ exists, it is unique and $(\mathbf{t}-\mathbf{a}^*)\cdot(\mathbf{a}-\mathbf{a}^*)\leq 0$ for each $\mathbf{a}$ in A (see Webster, [10] Theorem 2.4.1, p. 65). Moreover, the point $\mathbf{a}^*$ cannot be an *interior* point of A, because in this case, there exists in A some ball containing $\mathbf{a}^*$ which also contains other points of A different from $\mathbf{a}^*$ and nearer to $\mathbf{t}$. Thus, $\mathbf{a}^*$ is on the boundary of A. Then, there exists a hyperplane passing from $\mathbf{a}^*$ which supports A (see Webster, [10] Theorem 2.4.12, p. 71). Finally, we show that the hyperplane H passing from $\mathbf{a}^*$ and orthogonal to the line joining $\mathbf{t}$ and $\mathbf{a}^*$ is a support for A. Let: $h_0 + a_1h_1 + ... + a_Ch_C = 0$ be the equation of the hyperplane H, which can also be written: $h_0 + \mathbf{h}\cdot\mathbf{a} = 0$. We also have $h_i \geq 0$ for $i = 1, ..., C$, given that A is a comprehensive set. Suppose by contradiction that H does not support A. Then, for some vector $\hat{\mathbf{a}} \in A$, we have: $h_0 + \mathbf{h}\cdot$

$\hat{\mathbf{a}} > 0$, which gives: $\mathbf{h} \cdot (\hat{\mathbf{a}} - \mathbf{a}^*) > 0$. From the uniqueness of $\mathbf{a}^*$, we have: $(\mathbf{t} - \mathbf{a}^*)(\mathbf{a} - a^*) \leq 0$,$\forall \mathbf{a} \in \underline{A}$. Given that $\mathbf{t}$ is normal to H and external to it, we have $(\mathbf{t} - \mathbf{a}^*) = \lambda \mathbf{h}$, for some positive scalar λ. By substitution, we obtain $\mathbf{h} \cdot (\hat{\mathbf{a}} - \mathbf{a}^*) \leq 0$, a contradiction.

Proof of Lemma 2

Consider the hyperplane that supports A, perpendicular to the line joining $\mathbf{t}$ and $\mathbf{a}^* = (a_1^*,..,a_n^*)$ (the point of A with min distance from $\mathbf{t}$). Consider now an opportunity set H consisting of the part of this hyperplane lying in the positive orthant of $\mathbb{R}^C$. By construction, we have $H \supseteq A$. Axiom F guarantees the existence in H of at least C vectors $\tilde{x}^z = (\tilde{x}_1^z, ..., \tilde{x}_C^z)$, with $z = 1, ..., C$, such that $\tilde{x}_j^z > a_j^*$ if $j = z$ and $\tilde{x}_j^z < a_j^*$ if $j \neq z$.

We show now that $\mathbf{a}^*$ belongs to $co(\tilde{x}^1, ..., \tilde{x}^C)$, which implies $\mathbf{a}^* \in E(H)$.

Let $h_0 + h_1\tilde{x}_1 + ... + h_C\tilde{x}_C = 0$ be the equation of the hyperplane containing H and passing through a^*. This equation is satisfied by each point $\tilde{x}^z$.

In order to simplify the problem and without loss of generality, we consider the case $\mathbb{R}^3$. Applying a translation which puts the point $\mathbf{a}^* = (a_1^*, a_2^*, a_3^*)$ in the origin, we derive from Axiom F the existence in $E(H)$ of three vectors with coordinates:

$$\begin{aligned}
(x_1, y_1, z_1) &= (\tilde{x}_1^1 - a_1^*,\ \tilde{x}_2^1 - a_2^*,\ \tilde{x}_3^1 - a_3^*), \text{with } \ x_1 > 0,\ y_1 < 0,\ z_1 < 0 \\
(x_2, y_2, z_2) &= (\tilde{x}_1^2 - a_1^*,\ \tilde{x}_2^2 - a_2^*,\ \tilde{x}_3^2 - a_3^*), \text{with } \ x_2 < 0,\ y_2 > 0,\ z_2 < 0 \qquad (1) \\
(x_3, y_3, z_3) &= (\tilde{x}_1^3 - a_1^*,\ \tilde{x}_2^3 - a_2^*,\ \tilde{x}_3^3 - a_3^*), \text{with } \ x_3 < 0,\ y_3 < 0,\ z_3 > 0.
\end{aligned}$$

The equation of the hyperplane containing H becomes $xa + yb + zc = 0$, with $a, b, c > 0$.

We prove now that the origin can be expressed as a convex combination of the three points above. Using the equation of the hyperplane and denoting by (α, β, γ) the elements of the simplex Δ^2, we have just to show that the following homogenous system admits only positive solutions:

$$\begin{cases}
-\frac{y_1 b + z_1 c}{a}\alpha - \frac{y_2 b + z_2 c}{a}\beta - \frac{y_3 b + z_3 c}{a}(1 - \alpha - \beta) = 0 \\
y_1\alpha + y_2\beta + y_3(1 - \alpha - \beta) = 0 \\
z_1\alpha + z_2\beta + z_3(1 - \alpha - \beta) = 0.
\end{cases}$$

The solutions are:

$$\beta = \frac{y_3 z_1 - y_1 z_3}{y_2 z_3 - y_3 z_2 + y_3 z_1 - y_1 z_3 - y_2 z_1 + y_1 z_2},$$

$$\alpha = \frac{y_2 z_3 - y_3 z_2}{y_2 z_3 - y_3 z_2 + y_3 z_1 - y_1 z_3 - y_2 z_1 + y_1 z_2}.$$

Denoting $y_3 z_1 - y_1 z_3 = S$ and $y_2 z_3 - y_3 z_2 = R$, we obtain:

$$\begin{aligned}
\beta &= \frac{S}{R + S + y_1 z_2 - y_2 z_1} \\
\alpha &= \frac{R}{R + S + y_1 z_2 - y_2 z_1}
\end{aligned}$$

From $x_2 = -\frac{y_2 b + z_2 c}{a} < 0$ and $x_3 = -\frac{y_3 b + z_3 c}{a} < 0$, we find $R > 0$, and using conditions (1) we immediately obtain the result.

Finally, assume that $\mathbf{a}^*$ does not belong to $E(A)$. From $\mathbf{a}^* \in E(H)$ and $A \subseteq H$, we obtain a contradiction of Axiom C.

Proof of Theorem 1

It is easy to show that $\succsim_{\min}$ satisfies axioms I, E, C, S. Then we show that if $\succsim$ on $\mathbb{V}$ has essential sets which satisfy Axiom F and $\succsim$ satisfies Axioms I, E,C, S, then $\succsim = \succsim_{\min}$. Consider two opportunity sets A and $B \in \mathbb{V}$. Three case have to be considered.

1. Both opportunity sets contain the target. Then essential sets are not defined and only the inclusion axiom holds, so the proof is trivial.

2. The target $\mathbf{t}$ is exterior to A and to B. Let us denote respectively by $\mathbf{a}^*$ and $\mathbf{b}^*$ the points with minimal distance from the target of the opportunity sets A and B. Suppose by contradiction that $A \succsim B$, but $d(a^*, \mathbf{t}) > d(b^*, \mathbf{t})$. By Lemma 2, $a^* \in E(A)$ and $b^* \in E(B)$. As a consequence, for some scalar $\bar{\alpha} > 1$ we have $E(\bar{\alpha}A) \subseteq E(\bar{\alpha}B)$. Two sub cases are then possible.

   i) Suppose first that for some scalar $\bar{\alpha} > 1$,we have $E(\bar{\alpha}A) \subset E(\bar{\alpha}B)$. In this case, by inclusion axiom, $E(\bar{\alpha}B) \succ E(\bar{\alpha}A)$. Hence, by Axiom E and by transitivity we obtain $\bar{\alpha}B \succ \bar{\alpha}A$. Due to the axiom B, we can conclude, multiplying by $\frac{1}{\bar{\alpha}}$ both opportunity sets, that $B \succ A$, i.e. a contradiction.

   ii) Suppose now that for any scalars $\alpha > 1$ we cannot find the situation described in the previous point i). Let us multiply then both A and B by the min α such that $d(B, \mathbf{t}) = 0$. We denote $\hat{\alpha}$ such a scaler with $\hat{\alpha} > 1$ by construction. After this expansion of the opportunity set, the target $\mathbf{t}$ belongs to the boundary of $\hat{\alpha}B$. As a consequence, $\mathbf{t}$ coincides with the essential set of $\hat{\alpha}B$ and, we have $E(\bar{\alpha}A) \subseteq E(\bar{\alpha}B)$. Let us define now the new opportunity set H which is the positive portion of an hyperplane which supports $\hat{\alpha}B$ at the point $\mathbf{t}$. In this case we have that each vector of $E(\hat{\alpha}A)$, has all its elements less or equal to the elements of $\mathbf{t}$. This means that $E(\hat{\alpha}A)$ lies below H. More precisely, we have $E(\hat{\alpha}A) \subset H$ and, applying the inclusion axiom, we find $H \succ E(\bar{\alpha}A)$. Given that $E(H) = \mathbf{t} = E(\hat{\alpha}B)$ by using Axiom E, and transitivity we obtain $\hat{\alpha}B \succ \hat{\alpha}A$. By multiplying both sets by $\frac{1}{\hat{\alpha}}$ and by applying axiom S we find again the contradiction $\underline{B} \succ \underline{A}$.

3. If $\mathbf{t} \in A$ and $\mathbf{t} \notin B$, if $E(B) \subset A$, by applying set inclusion axiom we have $A \succ E(B)$ and by essentiality we immediately obtain $A \succ B$. On the other hand we always have, in this case, $d(A, \mathbf{t}) < d(B, \mathbf{t})$. If $E(B) \subseteq A$ we can repeat the reasoning of the point 2.ii) above in order to complete the proof.

Table 2: Health and education by households quintiles

Italy

	q1	q2	q3	q4	q5
E1-H1	**0,00**	**0,00**	**0,00**	**0,17**	**0,58**
E1-H1 N	0,00	0,00	0,00	0,08	0,25
E1-H1 C	0,00	0,00	0,00	0,08	0,08
E1-H1 S	0,00	0,00	0,00	0,00	0,25
E1-H2	**0,28**	**0,18**	**0,09**	**0,42**	**0,66**
E1-H2 N	0,00	0,09	0,09	0,08	0,25
E1-H2 C	0,19	0,00	0,00	0,17	0,25
E1-H2 S	0,09	0,09	0,00	0,17	0,17
E1-H3	**2,32**	**3,05**	**4,01**	**7,05**	**17,33**
E1-H3 N	0,65	0,81	1,31	2,38	7,39
E1-H3 C	0,65	0,54	1,31	1,53	4,73
E1-H3 S	0,93	1,70	1,40	3,14	5,23
E2-H1	**0,37**	**0,54**	**0,61**	**0,08**	**0,50**
E2-H1 N	0,37	0,27	0,26	0,00	0,25
E2-H1 C	0,00	0,09	0,09	0,08	0,17
E2-H1 S	0,00	0,18	0,26	0,00	0,08
E2-H2	**0,84**	**1,43**	**1,31**	**1,87**	**2,16**
E2-H2 N	0,19	0,27	0,52	0,85	1,25
E2-H2 C	0,19	0,45	0,35	0,59	0,33
E2-H2 S	0,47	0,72	0,44	0,42	0,58
E2-H3	**15,99**	**19,98**	**25,98**	**33,62**	**36,07**
E2-H3 N	4,19	5,56	11,62	14,70	22,18
E2-H3 C	2,42	6,46	6,38	8,92	8,47
E2-H3 S	9,40	8,07	8,03	9,94	5,48
E3-H1	**6,41**	**4,57**	**4,53**	**3,74**	**2,57**
E3-H1 N	1,68	1,43	2,53	2,46	1,58
E3-H1 C	1,77	1,35	1,05	1,02	0,58
E3-H1 S	2,98	1,70	0,96	0,25	0,42
E3-H2	**8,27**	**8,78**	**6,80**	**4,24**	**3,40**
E3-H2 N	1,68	2,87	3,14	1,70	1,99
E3-H2 C	1,58	2,15	1,48	1,36	0,91
E3-H2 S	5,03	3,77	2,18	1,19	0,50
E3-H3	**65,52**	**61,47**	**56,67**	**48,81**	**36,73**
E3-H3 N	13,59	19,73	23,67	27,53	23,09
E3-H3 C	12,76	13,27	16,16	13,25	9,39
E3-H3 S	39,20	28,43	16,77	8,07	4,15
total	**100**	**100**	**100**	**100**	**100**

Germany

	q1	q2	q3	q4	q5
E1-H1	**0,89**	**0,81**	**1,46**	**1,87**	**2,42**
E1-H1 E	0,41	0,24	0,73	0,49	0,56
E1-H1 W	0,49	0,57	0,73	1,38	1,85
E1-H2	**2,28**	**3,58**	**5,69**	**7,97**	**12,10**
E1-H2 E	0,81	1,71	2,76	2,60	1,61
E1-H2 W	1,46	1,87	2,93	5,37	10,48
E1-H3	**5,20**	**9,11**	**12,28**	**18,13**	**34,19**
E1-H3 E	1,30	4,15	4,80	5,45	5,73
E1-H3 W	3,90	4,96	7,48	12,68	28,47
E2-H1	**4,72**	**5,53**	**4,31**	**3,82**	**3,15**
E2-H1 E	1,63	1,95	1,54	1,06	0,24
E2-H1 W	3,09	3,58	2,76	2,76	2,90
E2-H2	**13,66**	**17,49**	**17,56**	**15,93**	**9,84**
E2-H2 E	6,83	7,65	7,72	4,55	1,13
E2-H2 W	6,83	9,85	9,84	11,38	8,71
E2-H3	**28,86**	**33,20**	**33,33**	**33,58**	**25,73**
E2-H3 E	12,20	12,86	11,79	7,80	3,39
E2-H3 W	16,67	20,34	21,54	25,77	22,34
E3-H1	**3,41**	**3,25**	**1,79**	**1,06**	**0,89**
E3-H1 E	0,41	0,90	0,65	0,08	0,00
E3-H1 W	3,01	2,36	1,14	0,98	0,89
E3-H2	**8,62**	**4,88**	**4,63**	**3,41**	**1,77**
E3-H2 E	1,87	1,46	0,65	0,57	0,08
E3-H2 W	6,75	3,42	3,98	2,85	1,69
E3-H3	**12,20**	**8,14**	**5,77**	**3,98**	**3,79**
E3-H3 E	2,68	1,38	0,73	0,33	0,00
E3-H3 W	9,51	6,75	5,04	3,66	3,79
n.a.	**20,16**	**14,00**	**13,17**	**10,24**	**6,13**
n.a. E	0,73	0,24	0,24	0,16	0,08
n.a. W	19,43	13,75	12,93	10,08	6,05
total	**100**	**100**	**100**	**100**	**100**

LEGEND

H: *Are you hampered by any physical or mental health problem, illness or disability?*

H1 = yes
H2 = yes, partially
H3 = no

E: *Highest level of education completed*

E1 = recognized 3rd level
E2 = second stage of secondary level
E3 = less than second stage